ISRAEL JOURNAL OF MATHEMATICS 112 (1999), 249–269

A MARKOV RANDOM FIELD WHICH IS K BUT NOT BERNOULLI

BY

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ABSTRACT

We use a variant of a discrete time exclusion process, studied by Yaguchi [7], to construct a Markov random field which is K but not Bernoulli. Instead of having all the particles in the exclusion process indistinguishable, this system has two different types of particles.

1. Introduction

Friedman and Ornstein proved that finite state space, mixing Markov chains are isomorphic to Bernoulli shifts [3]. In higher dimensions, however, this theorem does not hold. Ledrappier created a simple example of a $\mathbb{Z}^2$ Markov random field which is mixing, but is not 3-mixing, and has zero entropy [5]. Many other people have expanded upon Ledrappier's example to create a wide variety of zero entropy mixing $\mathbb{Z}^2$ Markov random fields. In this paper we construct a $\mathbb{Z}^2$ Markov random field which is K (equivalently, is of completely positive entropy, has trivial full tail [1] [4]) but is not isomorphic to a Bernoulli shift.

The example that we construct is the combination of two Markov random fields. The first is a simple zero entropy Markov random field (Ω', S, T, μ'). Let

$$\Omega' = (\omega' \in (\text{red, blue})^{\mathbb{Z}^2} \,|\, \omega'_{i,j} = \omega'_{i+1,j} \forall i, j)$$

$$\text{where } (S(\omega'))_{i,j} = (\omega')_{i,j+1} \text{ and } (T(\omega'))_{i,j} = (\omega')_{i+1,j}.$$

Received January 26, 1997

S and T are the shifts down and to the left and the measure μ' on vertical cylinder sets comes from the Bernoulli two shift. Consider the action of T as the progression of time. Since $T(\omega') = \omega'$ this system is not changing under time. This clearly prevents the process from being isomorphic to a Bernoulli shift. The second is a discrete time exclusion process on the integers introduced by Yaguchi [7]. This is a type of interacting particle system which we will show is isomorphic to a Bernoulli shift.

The example that we will show is K but not Bernoulli is what we call a colored exclsuion process. Each point is a pair (ω, z) where ω comes from the exclusion process and $z \in (\text{red, blue})^{\mathbb{Z}}$. The action of T will again represent the progression of time. The projection of $T(\omega, z)$ onto the second coordinate is either z or $\sigma(z)$, where σ is the shift. Thus for any time t the projection onto the second coordinate of $T^t(\omega, z)$ is a point of the form $\sigma^i(z)$. This is the key to showing that the colored exclusion process is not isomorphic to a Bernoulli shift.

The rest of the paper is organized as follows. In section 2 we define both the exclusion process and the colored exclusion process. In section 3 we prove that they are both Markov random fields. In section 4 we use Yaguchi's arguments to show that the exclusion process is K. Moreover, it is isomorphic to a Bernoulli shift. In section 5 we show that the exclusion process provides enough randomness so that colored exclusion process is K. Finally, in section 6 we prove that the colored exclusion process is not isomorphic to a Bernoulli shift. Moreover, it does not belong to a broader class of transformations called loosely Bernoulli transformations.

2. Construction

The basis of this example is a discrete time exclusion process on the integers which was originally studied by Yaguchi [7]. The description of it below is informal. See [7] for a more rigorous description. Some of the arguments in that paper have been reproduced here.

Arrange the integers in a vertical line, and on each integer place a container which can hold at most one particle. At any time the state of our system is given by $x \in X = (0,1)^{\mathbb{Z}}$, where $x_i = 1$ implies that there is a particle in the ith container. After one unit of time each of our particles will either stay in the same container or move down one container. To describe the movement of our particles we must choose α, $0 < \alpha \leq 1/2$. The process evolves in the following way. During each interval of time, every particle decides independently with probability α if it wants to move down one space. It also checks if the container below it is vacant. If both the particle decides to move and the container below is empty, then the particle moves down one container. Otherwise it stays in the same container.

The stationary measures on $(0,1)^{\mathbb{Z}}$ for this process were classified by Yaguchi. They are a family of point masses and a family indexed by α and ρ, $0 \le \rho \le 1$, which specifies a density of containers that have a particle. Define $x_m(i) = 0$ for all $i > m$ and $x_m(i) = 1$ for all $i \le m$. Then for each m the point mass at x_m is stationary. The following result of Yaguchi says for each α and ρ we have a unique stationary measure on X. Define $\gamma = (1/\rho) - 1$ and

$$\beta = \frac{\gamma + 1 - \sqrt{(\gamma+1)^2 - 4\gamma\alpha}}{2\gamma}.$$

LEMMA 1 (Yaguchi): *The nontrivial stationary measures for the exclusion process are the Markovian measures $\mu_{\alpha,\rho}$ given by the transition probabilities*

$$P(0,0) = 1 - \beta/\alpha,\ P(1,0) = \beta\gamma/\alpha,\ P(0,1) = \beta/\alpha,\ P(1,1) = 1 - \beta\gamma/\alpha$$

where $P(p,q) = \mu(x_{i+1} = q | x_i = p)$.

Proof: This is proved in [7]. Figure 1 gives a concrete example of a point in our space and of the measure $\mu_{\alpha,\rho}$. ∎

To convert this exclusion process into a two dimensional space-time process we use the space $\Omega = (0,1)^{\mathbb{Z}^2}$. S and T are the shifts down and to the left. Moving one column to the right corresponds with time increasing by one unit. The measure $\overline{\mu}_{\alpha,\rho}$ on Ω has measure $\mu_{\alpha,\rho}$ on vertical cylinder sets and is determined on other cylinder sets by $\mu_{\alpha,\rho}$ and the transition probabilities of the exclusion process. Now fix α and $0 < \rho < 1$ and drop the subscripts on μ and $\overline{\mu}$.

Now we color each particle independently so that the color of each particle is constant throughout time. This is illustrated in Figure 1. To do this coloring formally we use the exclusion process as a base for a skew product with the one dimensional Bernoulli two-shift.

We have already defined $\Omega = (0,1)^{\mathbb{Z}^2}$. Now define $Z = (\text{red, blue})^{\mathbb{Z}}$. Now, for the process we are interested in, $(\overline{S}, \overline{T}, \Omega \times Z, \overline{\mu} \times \nu)$, is defined as follows:

$$\overline{S}(\omega, z) = \begin{cases} (S(\omega), \sigma(z)) & \text{if } \omega_{0,0} = 1, \\ (S(\omega), z) & \text{if } \omega_{0,0} = 0, \end{cases}$$

and

$$\overline{T}(\omega, z) = \begin{cases} (T(\omega), \sigma(z)) & \text{if } \omega_{0,0} = 1 \text{ and } \omega_{1,0} = 0, \\ (T(\omega), z) & \text{else,} \end{cases}$$

where

$$(S(\omega))_{i,j} = (\omega)_{i,j+1} \quad \text{and} \quad (T(\omega))_{i,j} = (\omega)_{i+1,j}$$

are the shifts down and to the left and σ is the shift $(\sigma(z))_i = z_{i+1}$. The measure ν on Z comes from the Bernoulli two shift. The example in the introduction is the case when $\rho = 1$. We will work with the three set partition which tells us whether there is a ball in container 0 at time 0 and, if so, what color it is.

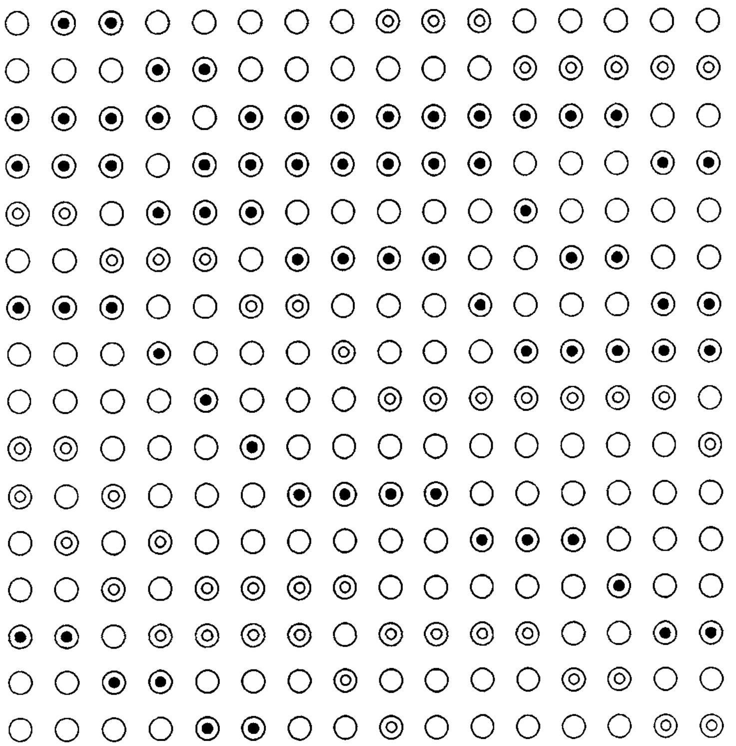

Figure 1. In this picture is a portion of a point $\omega \in \Omega$. A big circle represents a container, small circles and dots represent the two different types of colored particles. The flow of time is to the right. In the leftmost column is a portion of a point $x \in X$ which has

$$x_0, x_1, x_3, x_4, x_7, x_8, x_{10}, x_{14}, x_{15} = 0 \text{ and}$$

$$x_2, x_5, x_6, x_9, x_9, x_{11}, x_{12}, x_{13} = 1.$$

The $\mu_{\alpha,\rho}$ measure of all points that have these same values for x_0 through x_{15} is $(1-\rho)P(0,0)^4P(0,1)^4P(1,0)^4P(1,1)^3$.

3. The colored exclusion process is Markovian

Before we begin to show that the colored exclusion process is Markovian, we make a few definitions and observations about the exclusion process. A vertical cylinder set A is defined by a finite set $S_A \subset \mathbb{Z}$ and choices $A_j \in (0,1)$ for $j \in S_A$. Then $A = (x|x_j = A_j \forall j \in S_A)$. On vertical cylinders sets A we define the measure $\mu_{\mathbf{t}}(\mathbf{x})$ by

$$\mu_t(x) = \mu\{\omega \in T^t A|\ \omega_{0,i} = x_{0,i} \text{ for all } i \in \mathbb{Z}\}.$$

In other words $\mu_t(x)$ is the conditional probability the system is in set A at time t, given that it was in state x at time 0.

A right half plane cylinder set B is defined by a finite set $S_B \subset \mathbb{Z}^2$, $i \geq 0$ $\forall (i,j) \in S_B$ and choices $B_j \in (0,1)$. Then $B = (\omega|\omega_j = B_j \forall j \in S_B)$.

Definition 1: For right half plane cylinder sets, B, we define $\overline{\mu}(x)(B)$ to be the conditional probability the system is in set B given that it was in state x at time 0. The definition of the exclusion process gives us the following lemma.

LEMMA 2: *For a cylinder set B, $\overline{\mu}(x)(B)$ depends only on $x_m, ..., x_n$ where $m = \min j - i$, $(i,j) \in S_B$ and $n = \max j + i$, $(i,j) \in S_B$.*

Proof: This is easily proved by induction on the largest $i \in S_B$. ∎

The following lemma says that for finite periods of time the process develops independently in regions which are sufficiently far apart.

LEMMA 3: *If A and B are right half plane cylinder sets and there exists an m such that $j > m + i$ for all $i,j \in S_A$, $j < m - i$ for all $i,j \in S_B$, then*

$$\overline{\mu}(x)(A \cap B) = \overline{\mu}(x)(A)\overline{\mu}(x)(B).$$

Proof: Again this is proved by induction on the largest i. The lemma is vacuous for $i = 0$. The inductive step is true by applying Lemma 2 and the inductive hypothesis. ∎

The exclusion process is Markovian as a $\mathbb{Z}$ action and has a Markovian invariant measure. Now we will show that it is Markovian as a $\mathbb{Z}^2$ action.

Let R be a rectangle in $\mathbb{Z}^2$ and $\partial R = ((i,j)|\ d(i,j), R) = 1)$ (where $d((x,y),(x',y')) = \sup|x - x'|, |y - y'|)$. Now let B be a cylinder set defined on the boundary of R, I and J be cylinder sets defined inside and on the boundary of R, and O a cylinder set defined outside R such that O, I, and J all agree with B on the boundary of R.

Definition 2: A $\mathbb{Z}^2$ random field is **Markovian** if for any rectangle R, and cylinder set B, I, and O, $\overline{\mu}(I|B) = \overline{\mu}(I|B \cap O)$.

Remark 1: Often times the definition of a Markov random field is defined with the boundary of a rectangle using the L^1 metric,

$$d((x,y),(x',y')) = |x - x'| + |y - y'|.$$

While these two definitions are different, every Markov random field which satisfies our weaker definition has a generating partition which the satisfies the stronger definition.

LEMMA 4: *The exclusion process is Markovian.*

Proof: We will show that $\frac{\overline{\mu}(I|B\cap O)}{\overline{\mu}(J|B\cap O)}$ is independent of O, which is equivalent to our definition. Without loss of generality O can specify everything that happens on a big right isosceles triangle with hypotenuse a vertical line on the left hand side except inside the rectangle R. We can also choose I and J to specify everything that happens inside R and on the boundary of R. See Figure 2. Thus

$$\overline{\mu}(I \cap B \cap O) = \mu(L)\alpha^q(1-\alpha)^r \quad \text{and} \quad \overline{\mu}(J \cap B \cap O) = \mu(L)\alpha^m(1-\alpha)^n$$

for some q, r, m, and n, where L is the vertical cylinder set on the left hand side of the triangle.

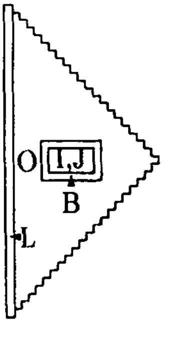

Figure 2. This illustrates the choice of the cylinder sets in the proof of Lemma 4.

Each of the above probabilities is the product of terms coming from either the left hand side, the movement of particles inside the rectangle or the movement of particles outside the rectangle. The terms from the left hand side and transitions outside the rectangle will cancel out in the ratio, leaving it independent of the choice of O. The following will make that precise.

Let

$$\begin{aligned} s_1 &= \#((i,j)|(i,j),(i+1,j-1) \in R \cup \partial R \text{ and } I_{(i,j)} = 1, I_{(i,j-1)} = 0, \\ &\qquad I_{(i+1,j)} = 1), \\ t_1 &= \#((i,j)|(i,j),(i+1,j-1) \in R \cup \partial R \text{ and } I_{(i,j)} = 1, I_{(i,j-1)} = 0, \\ &\qquad I_{(i+1,j)} = 0), \\ s_2 &= \#((i,j)|(i,j),(i+1,j-1) \in R \cup \partial R \text{ and } J_{(i,j)} = 1, J_{(i,j-1)} = 0, \\ &\qquad J_{(i+1,j)} = 0), \\ t_2 &= \#((i,j)|(i,j),(i+1,j-1) \in R \cup \partial R \text{ and } J_{(i,j)} = 1, J_{(i,j-1)} = 0, \\ &\qquad J_{(i+1,j)} = 1). \end{aligned}$$

Define o and p so that $o = q - s_1 = m - s_2$ and $p = r - t_1 = n - t_2$. This can be done since o and p are determined by values of O. Then

$$\overline{\mu}(I \cap B \cap O) = \mu(L)\alpha^{s_1}(1-\alpha)^{t_1}\alpha^{o}(1-\alpha)^{p}.$$

and

$$\overline{\mu}(J \cap B \cap O) = \mu(L)\alpha^{s_2}(1-\alpha)^{t_2}\alpha^{o}(1-\alpha)^{p}.$$

Thus

$$\frac{\overline{\mu}(I|B \cap O)}{\overline{\mu}(J|B \cap O)} = \frac{\overline{\mu}(I \cap B \cap O)}{\overline{\mu}(J \cap B \cap O)} = (\alpha)^{s_1-s_2}(1-\alpha)^{t_1-t_2}.$$

As this is independent of O, the exclusion process is Markovian. ∎

Lemma 5: *The colored exclusion process is Markovian.*

Proof: The colored exclusion process is Markovian because knowledge of the exclusion process on a rectangle and its boundary, and the coloring of the particles on the boundary determine the coloring of the particles inside of the rectangle. Thus knowledge of the coloring outside the rectangle and its boundary provides no additional information about the coloring inside the rectangle. It also provides no information about the exclusion process on the inside of the rectangle. Since the exclusion process is Markovian, the colored exclusion process is Markovian. ∎

4. The exclusion process is K

This section boils down to showing that if we know the particle system is in state $x \in (0,1)^{\mathbb{Z}}$ at time 0 then, for some large t, we know very little about where the particles are at time t. In order to show that the exclusion process is K as a $\mathbb{Z}^2$ action we will apply Conze's $\mathbb{Z}^2$ version, [1], of the Pinsker–Sinai–Rohlin theorem. The mth Conze tail is the subset of $\mathbb{Z}^2$, $((i,j)|\ i < -m$ or $i < 0$ and $j < -m)$.

THEOREM 1 (Conze): *A $\mathbb{Z}^2$ action is K if for any n and any $\epsilon > 0$ there exists an m and a set G, $\overline{\mu}(G) > 1 - \epsilon$, which is the union of cylinder sets defined in the mth Conze tail, and which has the following property. The conditional distribution of any cylinder set A defined inside $S = ((i,j)| - n \leq i, j \leq n)$ given any cylinder set $C \subset G$ is within ϵ of the unconditional distribution. That is, $|\overline{\mu}(A|C) - \overline{\mu}(A)| < \epsilon$ for all A defined in S and $C \subset G$ [1].*

In Figure 3, Conze's theorem says that conditioning on almost any cylinder set defined in regions 1,2, and 3 (the Conze tail) we get nearly the unconditional distribution in the square.

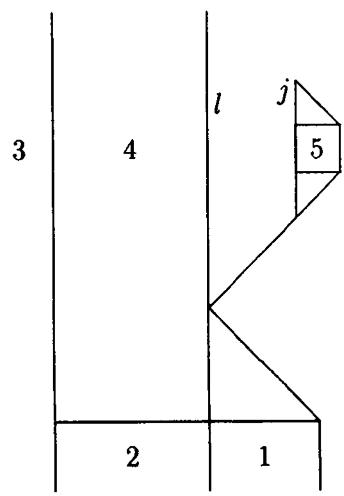

Figure 3. Conditioning on regions 1, 2, 3, and 4 tells us more than conditioning on the Conze tail, regions 1, 2, and 3. By the one dimensional Markov property, conditioning on regions 1, 2, 3, and 4 is the same as conditioning on the line l and region 1. By Lemma 3 the conditional distribution on region 5 given what happens on l and region 1 is the conditional distribution given what happens on

l. By Lemma 2 if the conditional distribution on vertical cylinder sets defined on the line segment j given l is near the unconditional distribution then the conditional distribution on region 5 given l is near the unconditional distribution. Thus the conditional distribution on region 5 given the Conze tail is near the unconditional distribution.

To start the proof that the exclusion process is K we make two observations which let us state a sufficient condition for the exclusion process to be K more easily. These are both illustrated in Figure 3.

Our first simplification is that, instead of conditioning on Conze tail pasts, we can condition on distant left half plane pasts. The caption of Figure 3 shows that conditioning on an appropriate Conze tail tells us no more than conditioning on a half plane past.

The second simplification is that, instead of showing that conditional distributions on a large square are close to the unconditional distribution, we only need to show that distributions on (larger) vertical cylinder sets are close to the unconditional distribution. By Lemma 2, the distribution on the (larger) vertical cylinder sets determines the distribution on the square. Thus the exclusion process is K as a $\mathbb{Z}^2$ action if for every l, vertical cylinder set A, with $S_A = [0, l]$, and $\epsilon > 0$ there exists a t such that for ϵ most x

$$|\mu_t(x)(A) - \mu(A)| < 2\epsilon,$$

where ϵ most x is used to mean there exists a set of x of measure $\geq 1 - \epsilon$.

In order to do this we introduce a coupling of the Markov process. The coupling creates a family of measures $\mu_t(x, y)$ on $X \times X$ which have marginals $\mu_t(x)$ and $\mu_t(y)$. If $x_i = y_i = 1$ then we say that the two particles are paired together. The idea of this coupling process is that it has particles that are paired together and move together as much as possible. Under this coupling we will show that the percentage of paired particles cannot decrease and tends towards one almost surely as time progresses.

The coupling works as follows. Suppose for some i there are particles in container i for both x and y, and they both are free to move ($x_i = y_i = 1$ and $x_{i-1} = y_{i-1} = 0$). Then the particles either move together ($x_i = y_i = 0$ and $x_{i-1} = y_{i-1} = 1$) with probability α, or stay together ($x_i = y_i = 1$ and $x_{i-1} = y_{i-1} = 0$) with probability $1 - \alpha$. If, for some i, the particle in container i in y is free to move, but the particle in container i in x cannot move ($x_i = y_i = 1, x_{i-1} = 1, y_{i-1} = 0$ and $x_{i-2} = 0$), then we cannot guarantee that the paired particles stay together. But we can demand that the particle in container i of y be paired with one of the particles in x. Thus we require with probability $1-2\alpha$ neither particle moves ($x_i = x_{i-1} = y_i = 1$), with probability α only the y particle moves ($x_i = x_{i-1} = 1, y_i = 0$), and with probability α only the x particle moves ($x_i = y_i = 1, x_{i-1} = 0$). In any other configuration there is no risk of breaking up paired particles without creating new pairs, so all other particles decide if they want to move independently. Since at least one of a pair of matched particles is still matched one instant later, the number of particles that are paired up does not decrease as time progresses.

Now we want to show that this coupling forces the fraction of particles that are paired to go to 1 as time progresses. Define $\rho(x,y)$, the density of disagreement of x and y, to be

$$\rho(x,y) = \lim_{n\to\infty} \frac{1}{2n+1}(\#\text{ of } i \in [-n,n] \text{ such that } x_i \neq y_i).$$

By the argument above, if the coupled system can evolve from (x,y) to (x',y') then $\rho(x,y) \geq \rho(x',y')$.

LEMMA (Yaguchi): *For every $p > 0$ there exists a t and $\delta > 0$ such that if $\rho(x,y) = \sigma > p$, and x and y are normal for μ, then*

$$\mu_t(x,y)((x',y')|\rho(x',y') < \sigma - \delta) = 1.$$

Proof: By the normality of x and y, and the density of disagreement, we can choose an l and $d > 0$ such that the density of N where there exists $i,j \in (Nl+1,(N+1)l)$ such that $x_i = 1$ and $y_i = 0$ and $x_j = 0$ and $y_j = 1$ is at least d. Consider one of those intervals. By time $t = l$ there is some positive probability ($> q = (\alpha(1-\alpha))^{l^2} > 0$) that the coupling process will have paired up at least two of the two previously unmatched particles that started out in the interval $(Nl, N(l+1)-1)$. By Lemma 3 the process develops independently in sufficiently separated regions. If we choose our intervals of length l at least $4l$ apart each will have independently probability $> q$ of pairing up at least one pair of previously unmatched particles. By the strong law of large numbers the coupled process will have almost surely eliminated at least a fixed fraction $\delta = dq/10l$ of the discrepancies by time t. Since the coupling does not allow the fraction of paired particles to decrease,

$$\mu_t(x,y)((x',y')|\rho(x',y') < \sigma - \delta) = 1. \quad \blacksquare$$

THEOREM: *The exclusion process is K.*

Proof: Repeated application of the previous lemma gives that for any ϵ there is a t large enough such that for almost all x and y,

$$\mu_t(x,y)((x',y')|\rho(x',y')<\epsilon)=1.$$

Given an l and an ϵ find t such that the comment above is true for ϵ^3/l. We say an interval, $(N, N+l-1)$, is good for x and y if, with probability $1-\epsilon$, the coupling makes no errors on this interval. Thus some interval, $(N, N+l-1)$, is good for ϵ^2 most (x,y). Thus for most x, most of the y make this interval good. Let A be a cylinder set specifying the location of particles in containers Nl to $(N+1)l-1$. Since $\mu(A)$ is the integral over all y of $\mu_t(y)(A)$, for these x, $|\mu_t(x)(A)-\mu(A)|<2\epsilon$. By translation invariance the same statement holds if A is a cylinder set specifying the location of particles in containers 0 to $l-1$. ∎

It is also possible to use the coupling to show that the exclusion process is isomorphic to a Bernoulli shift. This shows that the coloring is necessary to construct a Markov random field which is K but not Bernoulli.

5. The colored exclusion process is K

The previous section told us that if we know the exclusion process is in state x at time 0, then, for large t, this information tells us little about where the particles are at time t. Now, in order to show the colored exclusion process is K, we want to say that if we know the location and the colors of the particles at time 0, then, for large t, we know little about the location and colors of the particles in a finite number of containers at time t. We can make the same simplifications as in the previous section, so we only need to worry about the conditional distribution on vertical cylinder sets given half plane pasts.

There are three main steps for showing that the colored exclusion process is K. The first two are statements about the exclusion process only and the third is a statement about the exclusion process and colorings. First, we show that if we pick a typical point, $x \in X$ and a particle in x, and if we wait long enough, then the distribution of particles in a fixed number of containers below the container with our particle is almost the unconditional distribution. This is Lemma 8. Next we use that to prove that if we pick almost any particle in most any point at time 0, then, if we wait long enough, for every container the probability that our particle is in that chosen container at time t is small. This is Lemma 11. Finally, we will use this to show that if we condition on seeing a typical (x,z) at time 0, and any vertical cylinder set A telling us the location of particles in containers $-l$ to 0 at time t, this tells us very little about the color of those particles.

Define Ω^* to be the set of all points in Ω with a particle in container zero at time zero ($\Omega^* = (\omega|\ \omega_{0,0} = 1)$). Also let X^* to be the set of all states with a particle in container zero ($X^* = (x|\ x_0 = 1)$).

Given $x \in X^*$ we number the particles, calling the particle in container 0 to be particle 0, the first particle above particle 0, particle 1 and so on. We define x^i to be the container that particle i is in. Given $\omega \in \Omega^*$ we say that particle i is the particle which was the ith particle above 0 at time 0. Our goal is to show that if we pick a point $x \in X$, and a particle in x, and then wait long enough, we lose knowledge of the position of other particles in a fixed number of containers below our chosen particle. Since the dynamics of the process are shift invariant, we pick our point x and our particle, then shift x up or down so that our particle is in container 0. Then we study the behavior of the system given this new point. This way we only need to deal with $x \in X^*$.

Define the function f: $X^* \to \mathbb{N}^{\mathbb{Z}}$ so that $(f(x))_i = x^i - x^{i-1}$. $\mathbb{N}^{\mathbb{Z}}$ inherits a measure m from Ω^*. The function f is well defined for all points that have an infinite number of particles above and below zero, and is invertible on that set.

Define the following process on $\mathbb{N}^{\mathbb{Z}}$. For each $i \in \mathbb{Z}$ flip an $\alpha, 1-\alpha$ coin. If the α side is chosen and if $z_i > 1$, then decrease z_i by 1 and increase z_{i+1} by 1. Otherwise do nothing. Do this simultaneously for all i. This is a measure preserving process because it is essentially a restatement of our exclusion process. In this new process $z_i - 1$ represents the number of empty containers, "spaces", between particles i and $i-1$. If $x, y \in \mathbb{N}^{\mathbb{Z}}$ we say that $\min(x_i, y_i) - 1$ is the number of paired spaces between particles i and $i-1$ in the x name and the y name.

Now we will analyze this system and then pull our results back to the exclusion process. Define

$$P(x,y) = \lim_{n\to\infty} \frac{2\sum_{-n}^{n} \min(x_i, y_i)}{\sum_{-n}^{n} x_i + y_i}.$$

Next we introduce a coupling of this process. For each $i \in \mathbb{Z}$ use just one coin to determine if a space moves from i to $i+1$ in x and in y. This coupling defines a family of measures, $m_t(x,y)$, where $m_t(x,y)(S)$ is the conditional probability that the system was in state (x,y) at time 0 that it is in set S at time t. Under this coupling, if two spaces get paired up then they stay paired for all time.

Lemma 7: *For any $\epsilon > 0$ there exists a t such that for all normal x and y*

$$m_t(x,y)((x',y')|\ P(x',y') > 1-\epsilon) = 1.$$

Proof: First we prove that for any $\sigma < 1$ there exist a t and a δ such that for all normal x and y with $P(x,y) = p < \sigma$

$$m_t(x,y)((x',y')|\ P(x',y') > p+\delta) = 1.$$

As in the previous section, the lemma will be proven by repeated applications of the above statement.

By the normality of x and y and the density of disagreement, p, we can choose an l and $d > 0$ such that the density of N where there exists $i,j \in (Nl+1,(N+1)l)$ such that $x_i > y_i$ and $x_j < y_j$ is at least d. By time $t = l$ there is some positive probability ($> q = (\alpha(1-\alpha))^{l^2} > 0$) that the coupling process will have paired up the two previously unmatched spaces. For finite lengths of time the process develops independently in regions that are separated by large enough distances. So if we choose intervals, $(Nl+1,(N+1)l)$, that are at least $4l$ apart, each will have independently probability $> q$ of pairing up at least one pair of previously unpaired spaces. By the strong law of large numbers the coupled process will have almost surely paired up a fixed fraction $> qd/10l\rho = \delta$ of the unpaired spaces. Since paired spaces cannot be separated

$$m_t(x,y)((x',y')|P(x',y') > p+\delta) = 1. \qquad \blacksquare$$

Now we introduce a few definitions. These will let us make precise the statement that given most any particle in any point $x \in X^*$ at time 0, then by time t the distribution of particles in the l containers below the container particle 0 is almost the same as the unconditional distribution in X^*. Define

$$W_i^j(\omega) = \begin{cases} 0 & \text{if particle } i \text{ moves down between times } j \text{ and } j+1, \\ 1 & \text{if particle } i \text{ does not move down between times } j \text{ and } j+1. \end{cases}$$

For the rest of the paper A will be a vertical cylinder set which specifies where particles are in containers $-l$ to 0. Also let $\overline{A} = (\omega|\ \omega_{0,-l} = A_{-l}, \ldots, \omega_{0,0} = A_0)$.

Define

$$A_t = (\omega|\omega \in (T^{-t}S^{\sum_0^{t-1} W_j^0(\omega)}\overline{A}) \cap \Omega^*),$$

the set of all points in Ω^* such that at time t cylinder set A is directly below particle 0.

LEMMA 8: *Given any l and $\epsilon > 0$ there exists a t such that for most $x \in X^*$ and all $\overline{A}$*

$$|\overline{\mu}(x)(A_t) - \overline{\mu}(\overline{A}|\Omega^*)| < \epsilon.$$

Proof: Given ϵ choose N such that for ϵ^2 most x, we have $\sum_0^{l-1} x_i < N$. Also choose t such that by time t for ϵ^2 most x and y

$$m_t(x,y)((x',y')|\ P(x',y') > 1-(\epsilon/N)^2) = 1.$$

Then for ϵ^2 most points x and y that satisfy the second of those conditions

$$m_t(x,y)((x',y')|\ x'_i = y'_i \text{ for all } -l \leq i < 0) > 1-\epsilon.$$

Thus for most x, most y make this true. So for most x and all A

$$|\overline{\mu}(x)(A_t) - \overline{\mu}(\overline{A}|\Omega^*)| < \epsilon. \qquad \blacksquare$$

In the next two lemmas we show that if we start with a typical point $x \in X^*$ the paths that particle 0 takes from time T to time $T+M$ are very similar to an independent random walk. We will use this to show that for some time t, the conditional probability given that the system was in state x at time 0, that particle 0 is in container j at time t, is small for all j.

Define $P_b = (\omega|\omega \in \Omega^* \text{ and } W_0^j(\omega) = b_j\ \forall j,\ T \leq j < T+M)$.

LEMMA 9: *For any $\epsilon > 0$ there exists a T such that for most x*

$$\sum_{P_b} |\overline{\mu}_t(x)(P_b) - \overline{\mu}(P_b|\Omega^*)| < \epsilon.$$

Proof: This follows directly from Lemma 8 because the probability that a particle moves in a certain way from time 0 to n is determined by the particles in the $2n$ containers directly below our particle at time 0. $\blacksquare$

Now we quote a theorem from Yaguchi's paper.

LEMMA 10 (Yaguchi):

$$\overline{\mu}(P_b|\Omega^*) = \left(\frac{\alpha P(0,1)}{\gamma}\right)^i \left(1 - \frac{\alpha P(0,1)}{\gamma}\right)^{M-i}$$

where i is the number of j such that $b_j = 1$.

Proof: This is proved in [7]. $\blacksquare$

This next lemma says that given almost any $x \in X^*$ at time 0, we lose track of which container particle 0 is in as time progresses.

LEMMA 11: *For any $\epsilon > 0$ $\exists \overline{T}$ such that for most $x \in X^*$ and for all j*

$$\overline{\mu}(x)(\omega | \sum_0^{\overline{T}-1} W_i^0(\omega) = j) < \epsilon.$$

Proof: First we will sketch the proof, and then we will list the order to choose the parameters in a consistent manner. Choose T, M, and N so if $T \le a, b \le T + M$ and $b - a > N$, then for most x and all m

$$\overline{\mu}(x)(\omega | \sum_a^b W_0^i(\omega) = m) < \epsilon^2/4. \tag{1}$$

Either the lemma is true or there exists a time t_1 when the probability that the particle is in container c_1 is greater than $\epsilon/2$. Define

$$S_1 = (\omega | \, \omega_{0,j} = x_j \forall j, \sum_0^{t_1} W_0^j(\omega) = -c_1).$$

If there does exist such an S_1, consider the rest of the space $(\Omega \setminus S_1)$. If there is a time t_2 and container c_2 such that the probability that the particle was not in container c_1 at time t_1, but is in container c_2 at time t_2, is greater than $\epsilon/2$, then create S_2. Then apply the same procedure to the remaining space, $(\Omega \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1}))$, creating

$$S_i = \left(\omega | \, P(\omega) = x, \, \sum_0^{t_i} W_o^j(\omega) = -c_i, \, \omega \notin \bigcup_1^{i-1} (S_j) \right).$$

Continue in this manner until it is impossible to create another such S_i. There are at most $[2/\epsilon]$ ($[r]$ represents the greatest integer $\le r$) of these S_i since $\overline{\mu}(x)(S_i) > \epsilon/2$ and they are disjoint.

The points in S_i have their paths of particle 0 close together for times near t_i but not for times far from t_i. By equation (1), for any time t, $T \le t < t_i - N$ or $T + M \ge t > t_i + N$, a set of measure at most $\epsilon^2/4$ is in set S_i and particle 0 is in any given container at time t. Thus every time t and container c such that $t \notin (t_i - N, t_i + N)$ for all i at most $(\epsilon^2/4)[2/\epsilon] < \epsilon/2$ of the points in an S_i have particle 0 in container c at time t. By definition at most $\epsilon/2$ of the points in no S_i have particle 0 in container c at time t. Thus for those t no more than ϵ of the paths are in container c at time t.

Given ϵ, choose N such that for all i, $(\alpha P(0,1)/\gamma)^i (1 - \alpha P(0,1)/\gamma)^{N-i} < \epsilon^2/4$. Then choose a length of time M long enough so that $(2N + 1)[2/\epsilon]/M < \epsilon^2$.

Choose a distance T so that Lemma 9 holds with $\epsilon^2/8$ and M. Then for ϵ^2 most of the times from time T to time $T+M$ most particles have no container where more than ϵ of the paths are in any given container. Thus there exists a time $\overline{T}$ between T and $T+M$ where most points $x \in X^*$ have no container, c, where the conditional probability given x that particle 0 is in container c at time $\overline{T}$ is greater than ϵ. ∎

The previous lemma says that given most any starting point x, then if we wait long enough we lose track of where particle 0 is. Theorem 2 says that given most any staring point x we lose track of the position of particles over time. The next lemma is dedicated to showing that, as time progresses, we lose knowledge of these two things simultaneously.

That is for R sufficiently large, conditioning on starting point x at time 0, and that there is a particle in container i at time R, then we don't know where that particle was at time 0. Furthermore, if we condition on seeing x at time 0 and A at time R, then we still don't know where the particles we are seeing in A at time R were at time 0. Thus knowing the coloring at time 0 will not tell us much about the coloring of the particles in $(-l,0)$.

To make this precise we define $D^{i,R}(m)$ to be the conditional probability given we see x at time zero and a particle in container i at time R that it is particle m. Thus

$$D^{i,R}(m) = \frac{\overline{\mu}(x)(\omega|\ \sum_{j=0}^{R-1} W_m^j(\omega) = x^m - i)}{\overline{\mu}(x)(\omega|\ \omega_{R,i} = 1)}.$$

We also need to introduce another definition. We are concerned only with A that indicates the presence of at least one particle. For those A that don't, the coloring is not noticeable. Let k be the highest container that A says must have a particle. Given x and A, we define a distribution $D_A^{i,R}$. This distribution tells us the conditional probability given that we see x at time 0 and $T^{-i}A$ at time R that the particle m is the highest numbered particle in or below container i:

$$D_A^{i,R}(m) = \frac{\overline{\mu}(x)(\omega \in T^{-i}A,\ \sum_0^{R-1} W_m^j(\omega) = x^m - i - k)}{\overline{\mu}(x)(\omega|\ \omega \in T^{-i}(A))}.$$

LEMMA 12: *For any $\epsilon > 0$ there exist R such that for most x for most i and for all m, $D^{i,R}(m) < \epsilon$ and $D_A^{i,R}(m) < \epsilon$.*

Proof: By Theorem 2 we know that for most x the denominators are approximately ρ and $\mu(A)$ respectively. Thus it suffices to show that numerators can be made arbitrarily small. If A indicates there is a particle in container k, $\overline{\mu}(x)(\omega|\ \sum_{j=0}^{R-1} W_m^j(\omega) = x^m - i - k)$ is an upper bound on the numerators.

The previous lemma holds for a given particle based on the location of particles in a finite number of containers below the given particle. Thus for a normal point x most particles satisfy Lemma 11. Choose R big enough so that Lemma 11 holds with ϵ^2. For each i and m such that $\overline{\mu}(x)(\omega| \sum_{j=0}^{R-1} W_m^j(\omega) = x^m - i) > \epsilon$ means that Lemma 11 did not hold for particle m. Each particle m for which Lemma 11 does not hold can only lead to at most $[1/\epsilon]$ such i. Since Lemma 11 holds for ϵ^2 most m there are at most $\epsilon^2[1/\epsilon]$ i such that for any m, $\overline{\mu}(x)(\omega| \sum_{j=0}^{R-1} W_m^j(\omega) = x^m - i) > \epsilon$. Since the numerators can be made to be arbitrarily small, the lemma is true. ∎

Now we use Lemma 12 and the weak law of large numbers and condition on seeing (x, z) at time 0, and A at time t. We will show that this does not tell us much about the coloring of the particles in containers $-l$ to 0 at time t. This gives us the following theorem.

THEOREM 3: *The colored exclusion process is K.*

Proof: This proof is an adaptation of Mejilson's theorem [6] that a suitable skew product of a Bernoulli shift with an ergodic transformation is K. We will condition on having the state of the colored exclusion process at time 0 be (x, z), and the location of particles in containers $-l$ to 0 at time t be A. This gives us a distribution on the possible colorings of the particles. We break this distribution up into several parts. On each of these we apply the weak law of large numbers. Then, when we sum up over all the parts, we see that the conditional distribution of colorings is nearly the unconditional distribution of colorings.

By the same arguments as in the previous section we only need to worry about the probability of vertical cylinder sets when conditioning on half-plane pasts. Let $C = (z|\ z_{-l} = c_{-l}, \ldots, z_0 = c_0)$. Thus $\nu(C) = 1/2^{l+1}$. (C typically provides enough information to color more particles than are in containers $-l$ to 0.) Define $(\mu \times \nu)_t(x, z)(S)$ to be the conditional probability that we see S at time t given x was the state of the exclusion process at time 0 and z was the coloring. Our condition for the colored exclusion process to be K is that for any $\epsilon > 0$ there exists a t such that for 10ϵ most (x, z) and for all A and C

$$|(\mu \times \nu)_t(x, z)(A \cap C) - \mu \times \nu(A \cap C)| < 10\epsilon.$$

Fix l, ϵ, A and C. Find a p such that the average of p independent random variable that take value 1 with probability $1/2^{l+1}$, and 0 with probability $1 - 1/2^{l+1}$, is within ϵ of $1/2^{l+1}$ with probability at least $1 - \epsilon^2$. Choose γ small enough so that $p\gamma(l + 1) < \epsilon$. Choose a t large enough so that the previous lemma holds with γ. Fix i and now choose t such that Lemma 12 holds with γ and Theorem 2 holds with ϵ. Then choose δ so that $t\delta < \epsilon$. Now that i, t, and A have been fixed we drop the subscripts and superscripts on our $D(j)$. Let $D_\delta(j) = [D(j)/\delta]\delta$.

Divide D_δ into pieces D_n such that

1. $\sum_n D_n(j) \leq D_\delta(j)$ for all i,
2. $D_n(j) = \delta$ or 0,
3. $\sum_j D_n(j) = p\delta$, and
4. $D_n(j) > 0$ and $D_n(k) > 0$ implies $|k - j| > l + 1$.

By the paragraph above and our conditions on our D_n, if we do this optimally we have that

$$\sum_j |D(j) - D_\delta(j)| < t\delta < \epsilon \quad \text{and} \quad \sum_j |D_\delta(j) - \sum_n D_n(j)| << (l+1)p\gamma < \epsilon.$$

The first is true because $D(j) > 0$ for at most t different j. The second is true because the summand is less than γ and is greater than zero for at most $p(l+1)$ different j.

Let S_n be the set of k for which $D_n(k) = \delta$. For each n, the sets $(z|B^k(z) \in C)$ for $k \in S_n$ are independent. We now apply the weak law of large numbers to conclude that

$$\nu\left(z\middle|\left|\sum_{k\in S_n} \chi_C(B^k(z)) - \frac{|S_n|}{2^{l+1}}\right| < \epsilon^2|S_n|\right) > 1 - \epsilon^2.$$

Summing over n and multiplying by δ yields

$$\nu\left(z\middle|\left|\sum_k D_\delta(k)\chi_C(B^k(z)) - \frac{1}{2^{l+1}}\sum_k D_\delta(k)\right| < 2\epsilon\right) > 1 - 2\epsilon$$

and thus

$$\nu\left(z\middle|\left|\sum_k D(k)\chi_C(B^k(z)) - \frac{1}{2^{l+1}}\right| < 3\epsilon\right) > 1 - 2\epsilon.$$

By definition,

$$\mu \times \nu(A \cap C) = \mu(A)\nu(C) = \mu(A)\frac{1}{2^{l+1}}.$$

For 5ϵ most (x, z)

$$|\mu \times \nu_t(x,z)(A \cap C) - \mu \times \nu(A \cap C)| = \left|\mu_t(A)\sum_k D(k)\chi_C(B^k(z)) - \mu_t(A)\frac{1}{2^{l+1}}\right|$$
$$\leq 5\epsilon.$$

By Lemma 12 we can do this procedure for most x. By Theorem 2, for most x, $|\mu_t(x)(A) - \mu(A)| < \epsilon$. Thus we have that for 10ϵ most (x, z) are such that

$$|\mu \times \nu_T(x, z)(A \cap C) - \mu \times \nu(A \cap C)| < 10\epsilon.$$

Thus the colored exclusion process is K. ∎

COROLLARY 1: *The colored exclusion process has a trivial full tail.*

Proof: By the theorem of den Hollander and Steif every $\mathbb{Z}^2$ Markov random field which is K has a trivial full tail [4]. ∎

6. The colored exclusion process is not loosely Bernoulli

The simple zero entropy Markov random field mentioned in the introduction is not loosely Bernoulli. It is not because an ϵ good $\overline{f}$ matching of two arbitrary names in the space would imply for every $\epsilon > 0$ there is an ϵ good $\overline{f}$ of two arbitrary sequences of 0s and 1s. Although the arguments that follow are more complicated, the idea behind them is the same.

THEOREM 4: *The colored exclusion process is not isomorphic to a Bernoulli shift.*

Proof: Because the colored exclusion process is Markovian, by the theorem of den Hollander and Steif [4], we only need to show the colored exclusion process is not **Følner independent**. The definition of Følner independent is the same as of very weak Bernoulli, except instead of conditioning on the past tail we condition on the full tail. More precisely, a $\mathbb{Z}^2$ process is **Følner independent** if for every $\epsilon > 0$ there exists an N and a set G of measure at least $1 - \epsilon$ so that for any point x

$$\bar{d}_{[0,N]\times[0,N]}(\mu, \mu_{x,N}) < \epsilon,$$

where $\mu_{x,N}(A) = \mu(\omega \in A |\ \omega_{i,j} = x_{i,j} \text{ for all } (i,j) \notin \{0,N\} \times \{0,N\})$

There is an $\epsilon' > 0$ and a T, such that for $t > T$ any small translate ($\leq 4t$) of any name a', from the Bernoulli two shift, is $\overline{f}_t$ more that ϵ' separated from any small translate of more than half of the other points. Define

$$S(a') = (c' |\ \overline{f}_t(B^i(c'), B^j(a')) > \epsilon'\ \forall 0 \leq i, j \leq 4t).$$

Thus $\mu(S(a')) > .5$ for all a'. Suppose that we have an N such that the colored exclusion process is Følner independent with N and $\epsilon = \rho\epsilon'/100$, and $.5\rho N > T$.

An N past is any cylinder set defined in the full N tail, $(i,j)|(i,j \notin (0,N))$. An N past is good if every $(a,a') \in A$ has at least $.2\rho N$ particles in containers 0 through N for all times from 0 to N. We also require that the conditional distribution on this square given A is ϵ' $\overline{d}$ close to the unconditional distribution. The ergodic theorem and the assumption that the colored exclusion process is Følner independent tell us that most pasts are good.

We will show more than that most good pasts A and C give substantially different conditional distributions on the square. For most A and C there are no two points $(a,a') \in A$ and $(c,c') \in C$ that are close in $\overline{d}$. In fact there are no two points $(a,a') \in A$ and $(c,c') \in C$ that at one time are close in $\overline{d}$.

If the colored exclusion process is Følner independent, then there exist good pasts A and C and points $(a,a') \in A$ and $(c,c') \in C$ with $c' \in S(a')$, and some t such that there is an ϵ good $\overline{d}$ matching at time t of (a,a') with (c,c').

The $\overline{d}$ matching on column t induces an $\overline{f}$ matching of the colorings in the following way. Write down the ordered pair (l,j) if there exists a k such that particle l of (a,a') is in container k at time t, particle j of (c,c') is in container k at time t, and both particles have the same color.

Now we check that this forms an $\overline{f}$ matching. If there is an i, greater than the minimum i in the list and less than the maximum i in the list, for which there is no pair (i,j), then there must have been an error in the $\overline{d}$ matching. The same is true for j and the pairing is monotonic. Since the $\overline{d}$ matching is good these ordered pairs form a good $\overline{f}$ matching from a small translate of a' to a small translate of c'. Thus $c' \notin S(a')$ and we have a contradiction. ∎

Finally, we show that the colored exclusion process is not loosely Bernoulli.

THEOREM 5: *The colored exclusion process is not loosely Bernoulli.*

Proof: To show that it is not loosely Bernoulli we need to repeat essentially the same procedure. The difference is that we will arrive at a contradiction by having one time, and the image of that time under an $\overline{f}$ matching close. Choose the good pasts A and C, points (a,a') and (c,c') with $c' \in S(a')$ as in the previous section. Also choose f,

$$f\colon (0,N) \times (0,N) \to (0,N) \times (0,N), f(a,b) = (f_1(a,b), f_2(a,b)),$$

such that f is an ϵ good $\overline{f}$ matching of (a,a') and (c,c'). That is

$$\frac{1}{N^2}[\# \text{ of } ((i,j)|(a,a')_{(i,j)} \neq (c,c')_{f(i,j)})+$$

$$+\sum |f_1(i,j) - f_1(i+1,j) + 1| + |f_2(i,j) - f_2(i,j+1) + 1|] < \epsilon.$$

Choose column i of (a, a') so that it and its image under f in (c, c') are $\overline{f}$ close. Write down the ordered pair (l, j) if there exists a k such that particle l of (a, a') is in container k at time t and particle j of (c, c') is in container $f_1(t, k)$ at time $f_2(t, k)$ and both particles have the same color.

If there is an i, greater than the minimum i in the list and less than the maximum i in the list, for which there is no pair (l, j), then there must have been an error in the $\overline{f}$ matching. If the pair (l, j) is not followed by the pair $(l+1, j+1)$, then there also must have been an error or a time when $f_1(l,j) - f_1(l+1,j) \neq 1$ or $f_2(l,j) - f_2(l,j+1) \neq 1$. If there is a pair (l, j) and also a pair (l', j), then there also must have been a time when

$$f_1(l,j) - f_1(l+1,j) \neq 1 \quad \text{or} \quad f_2(l,j) - f_2(l,j+1) \neq 1.$$

Eliminate these pairs of the form (l, j) and (l', j) from the list.

Thus these remaining ordered pairs form a good $\overline{f}$ matching from a small translate of a' to a small translate of c'. Thus $c' \notin S(a')$ and we have a contradiction. ∎

ACKNOWLEDGEMENT: The author thanks Zohar Yakini for helpful discussions, and Don Ornstein for guidance on this portion of my thesis.

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